<u>cos 10° cos 30° cos 50° cos 70°</u>

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1. Product sum formula

Analysis :

(a) cos 30° is connected with special angle and should be singled out.

(b) You can then combine any two of the three : cos 10° , cos 50°, cos 70° to form a product. Then change this to a sum.

Steps :

cos 10° cos 30° cos 50° cos 70° = cos 30° cos 70° (cos 50° cos 10°)

$$= \frac{\sqrt{3}}{2}\cos 70^\circ \times \frac{1}{2}\left(\cos 60^\circ + \cos 40^\circ\right) = \frac{\sqrt{3}}{4}\cos 70^\circ \times \left(\frac{1}{2} + \cos 40^\circ\right)$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{2}\cos 70^\circ + \cos 70^\circ \cos 40^\circ\right] = \frac{\sqrt{3}}{4}\left[\frac{1}{2}\cos 70^\circ + \frac{1}{2}\left(\cos 110^\circ + \cos 30^\circ\right)\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{1}{2}\cos 70^\circ + \frac{1}{2}\left(-\cos 70^\circ + \frac{\sqrt{3}}{2}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{\sqrt{3}}{4}\right] = \frac{3}{16}$$

2. Vieta's theorem

Analysis :

(a) We should think 'backward'. Let A = 10°. It is not 'special'. However, 3A = 30° is special.

So $\cos 3A = \frac{\sqrt{3}}{2}$ and we may think of solving the equation : $2\cos 3A - \sqrt{3} = 0$

(b) The given is a product and we therefore need the Vieta's theorem, which can relate the product of roots with the coefficients of the equation. Note that in the calculation below, we get only a cubic equation.

Steps :

(a) $2\cos 3A - \sqrt{3} = 0 \quad \Rightarrow \quad \cos 3A = \frac{\sqrt{3}}{2}$

$\Rightarrow$ 3A = 30°, 330° , 390° (other roots neglected) $\Rightarrow$ A = 10°, 110° , 130°

(b) $2\cos 3A - \sqrt{3} = 0 \quad \Rightarrow \quad 2\left(4\cos^3 A - 3\cos A\right) - \sqrt{3} = 0$

$\Rightarrow \quad 8\cos^3 A - 6\cos A - \sqrt{3} = 0$

$\Rightarrow$ x = cos A is a root of $8x^3 - 6x - \sqrt{3} = 0$

(c) x = cos 10°, cos 110° cos 130° are roots of $8x^3 - 6x - \sqrt{3} = 0$.

By Vieta's Theorem, $\cos 10^\circ \cos 110^\circ \cos 130^\circ = -\frac{\text{coeff. of constant term}}{\text{coeff. of } x^3 \text{ term}} = \frac{\sqrt{3}}{8}$

$$\cos 10^\circ\left(-\cos 70^\circ\right)\left(-\cos 50^\circ\right) = \frac{\sqrt{3}}{8}$$

$$\therefore \quad \cos 10^\circ \cos 50^\circ \cos 70^\circ = \frac{\sqrt{3}}{8}$$

$$\therefore \quad \cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{\sqrt{3}}{8}\frac{\sqrt{3}}{2} = \frac{3}{16}$$

3. Complex number

Analysis :

(a) Naturally, we like to begin with $z = \cos 10° + i \sin 10°$ and obviously we have to use de Moivres' theorem, since $z^3 = \cos 30° + i \sin 30°$, $z^5 = \cos 50° + i \sin 50°$, $z^7 = \cos 70° + i \sin 70°$.

(b) During the calculation in part (b) below, we find that it is rather difficult to simplify. We therefore need more relations to 'lower' the power. Equations (1) and (2) are then found.

Steps :

(a) $z = \cos 10° + i \sin 10° \quad \Rightarrow \quad 1/z = \cos 10° - i \sin 10°$

$$\cos 10^\circ = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}, \qquad \cos 30^\circ = \frac{1}{2}\left(z^3 + \frac{1}{z^3}\right) = \frac{z^6+1}{2z^3}$$

$$\cos 50^\circ = \frac{1}{2}\left(z^5 + \frac{1}{z^5}\right) = \frac{z^{10}+1}{2z^5}, \qquad \cos 70^\circ = \frac{1}{2}\left(z^7 + \frac{1}{z^7}\right) = \frac{z^{14}+1}{2z^7}$$

(b) $$\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{z^2+1}{2z} \times \frac{z^6+1}{2z^3} \times \frac{z^{10}+1}{2z^5} \times \frac{z^{14}+1}{2z^7} = \frac{(z^2+1)(z^6+1)(z^{10}+1)(z^{14}+1)}{16z^{16}}$$

(c) It is difficult to simplify (b). But note that :

$$z = \cos 10° + i \sin 10° \quad \Rightarrow \quad z^9 = \cos 90° + i \sin 90° \quad \Rightarrow \quad z^9 = i \qquad .... \quad (1)$$

$$z^3 = \cos 30^\circ + i \sin 30^\circ = \frac{\sqrt{3}}{2} + \frac{1}{2}i,$$

$$z^6 = \cos 60^\circ + i \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}i = i\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) + 1 \quad \Rightarrow \quad z^6 = iz^3 + 1 \qquad .... \quad (2)$$

(d) Substitute (1) in part (b),

$$\cos 10^\circ \cos 30^\circ \cos 50^\circ \cos 70^\circ = \frac{(z^2+1)(z^6+1)(iz+1)(iz^5+1)}{16iz^7} = \frac{(z^2+1)(z^6+1)(-z^6+iz^5+iz+1)}{16iz^7}$$

$$= \frac{(z^2+1)(z^6+1)(-iz^3-1+iz^5+iz+1)}{16iz^7} \quad , \quad \text{by (2)}$$

$$= \frac{(z^2+1)(z^6+1)(-iz^3+iz^5+iz)}{16iz^7} = \frac{(z^2+1)(z^6+1)(-z^2+z^4+1)}{16z^6}$$

$$= \frac{(z^6+1)\left[(z^2+1)(z^4-z^2+1)\right]}{16z^6}$$

$$= \frac{(z^6+1)(z^6+1)}{16z^6} = \frac{1}{4}\left(\frac{z^6+1}{2z^3}\right)^2$$

$$= \frac{1}{4}(\cos 30^\circ)^2 = \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)^2 = \underline{\underline{\frac{3}{16}}}$$